ON BOOLEAN ALGEBRAS OF PROJECTIONS AND ALGEBRAS OF OPERATORS([1])

BY

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1. Introduction. This paper is concerned with certain problems involving
Boolean algebras of projections which arise when one attempts to extend the
theory of self-adjoint and normal operators to a general Banach space (the
theory of spectral operators; cf. [7; 8; 1]). The first part of this paper con-
tains an investigation of the notion of completeness for Boolean algebras of
projections in a Banach space. The results are then applied to investigate
operator algebras generated by projections.

Let $\mathfrak{B}$ denote a Boolean algebra (B.A.) of projections in a real or complex
Banach space. The weakest concept of completeness for $\mathfrak{B}$ is that $\mathfrak{B}$ should
be complete as an abstract Boolean algebra; that is, a least upper bound
$\bigvee E_\alpha$ shall exist for every subset $\{E_\alpha\} \subseteq \mathfrak{B}$. A more useful requirement is that,
in addition, the manifolds $\{E_\alpha \mathfrak{X}\}$ should span $(\bigvee E_\alpha)\mathfrak{X}$. In this latter case
we shall say $\mathfrak{B}$ is complete. Correspondingly, we have the notion of σ-com-
pleteness for $\mathfrak{B}$. §2 contains preliminary results relative to completeness. It
is shown that if $\mathfrak{B}$ is σ-complete as an abstract Boolean algebra, then $\mathfrak{B}$ is
bounded, i.e., $|E| \le M$, $E \in \mathfrak{B}$. If $\mathfrak{B}$ is σ-complete then the closure $\overline{\mathfrak{B}}^s$ of $\mathfrak{B}$
in the strong operator topology is a complete Boolean algebra of projections.
In §3 an important tool of the paper is developed. Regarding a σ-complete
B.A. $\mathfrak{B}$ of projections as a spectral measure defined on its Stone representation
space, it is shown (Theorem 3.1) that for each $x_0 \in \mathfrak{X}$, there exists a linear
functional x_0^* in $\mathfrak{X}^*$ such that the measure $x_0^* E(\cdot)x_0$ is positive, and the
countably additive vector valued measure $E(\cdot)x$ is absolutely continuous
with respect to the scalar measure $x_0^* E(\cdot)x_0$. This generalizes the situation for
self-adjoint projections in Hilbert space where one may take $x_0^* E(\cdot)x_0$
$=(E(\cdot)x_0, x_0)$. As an important consequence of the existence of x_0^* we obtain
the fact that if a directed system $\{E_\alpha\}$ of projections in $\mathfrak{B}$ converges weakly
to a projection E_0, then it converges strongly, i.e., $\lim_\alpha E_\alpha x = E_0 x$, $x \in \mathfrak{X}$.

In §4 we study the uniformly and weakly closed algebras which are gener-
ated by a bounded B.A. of projections. If $\mathfrak{B}$ is σ-complete, then the weakly
closed algebra generated by $\mathfrak{B}$ is the uniformly closed algebra generated by
the strong closure $\overline{\mathfrak{B}}^s$ of $\mathfrak{B}$. If in addition $\mathfrak{B}$ has simple spectrum (cf. §4),

Presented to the Society, October 30, 1954 and December 28, 1954 under the titles *On
completeness of Boolean algebras of projections in Banach spaces* and *On functions of spectral
measures*; received by the editors January 4, 1954.

([1]) The research contained in this paper was done under Contract onr 609(04) with the
Office of Naval Research.

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then the weakly closed algebra generated by $\mathfrak{B}$ equals the commutant of $\mathfrak{B}$. These results extend well known theorems in Hilbert space theory. An application is made to the theory of spectral operators to show that every operator in the weakly closed algebra generated by a scalar type spectral operator of class $\mathfrak{X}^*$ is again of this type.

Certain of the results of §§2 and 4 extend to general Banach spaces results proved in [1] for reflexive spaces by other methods.

2. **Completeness.** A *Boolean algebra of projections* in a Banach space $\mathfrak{X}$ is a set $\mathfrak{B}$ of commuting projections containing 0 and I which is a Boolean algebra (B.A.) under the operations

$$E \vee F = E + F - EF, \qquad E \wedge F = EF.$$

Thus $\mathfrak{B}$ is a lattice under the partial ordering in which $E \leq F$ if and only if $E \wedge F = E$. Equivalently, $E \leq F$ if and only if $E\mathfrak{X} \subseteq F\mathfrak{X}$, since E and F commute. It is easily seen that $(E \vee F)\mathfrak{X} = \text{clm } (E\mathfrak{X}, F\mathfrak{X})$ and $(I - (E \vee F))\mathfrak{X} = (I - E)\mathfrak{X} \cap (I - F)\mathfrak{X}$. A B.A. $\mathfrak{B}$ is *bounded* if there is a constant M such that $|E| \leq M$, $E \in \mathfrak{B}$.

In this section we introduce various notions of completeness for a B.A. of projections. The weakest of these from the standpoint of its connection with the topology of $\mathfrak{X}$ is the purely algebraic condition that $\mathfrak{B}$ should be complete (or σ-complete) as an *abstract* Boolean algebra. This means that for each subset (sequence) $\{E_\alpha\} \subseteq \mathfrak{B}$ there exist projections $\vee E_\alpha$ and $\wedge E_\alpha$ in $\mathfrak{B}$ which are respectively the least upper bound and greatest lower bound of $\{E_\alpha\}$ under the ordering $\leq$ in $\mathfrak{B}$. The strongest notion of completeness requires in addition that the manifolds $\{E_\alpha \mathfrak{X}\}$ should span $(\vee E_\alpha)\mathfrak{X}$. Because of its importance we shall call this merely "completeness."

2.1. DEFINITION. A B.A. of projections $\mathfrak{B}$ is *complete* (σ-*complete*) if for each subset (sequence) $\{E_\alpha\} \subseteq \mathfrak{B}$,

(a) $\mathfrak{X}$ admits the direct sum decomposition $X = \mathfrak{M} \oplus \mathfrak{N}$ where([2])

$$\mathfrak{M} = \text{clm } \{E_\alpha \mathfrak{X}\}, \qquad \mathfrak{N} = \bigcap_\alpha (I - E_\alpha)\mathfrak{X};$$

and

(b) the projection E_0 with range $\mathfrak{M}$ and null manifold $\mathfrak{N}$ defined by this decomposition belongs to B.

In this definition clearly $E_0 = \vee_\alpha E_\alpha$. Also a complete B.A. of projections is complete as an abstract B.A. Using Lemma 2.5 below one may easily construct examples to show the converse need not be true. Such an example is given following Corollary 3.3. It follows from the relation $\wedge_\alpha E_\alpha = I - \vee_\alpha(I - E_\alpha)$ that in a complete B.A. $(\wedge_\alpha E_\alpha)\mathfrak{X} = \bigcap_\alpha E_\alpha \mathfrak{X}$ and $(I - \wedge_\alpha E_\alpha)\mathfrak{X} = \text{clm } \{(I - E_\alpha)\mathfrak{X}\}$.

A net $\{E_\alpha\}$, $\alpha \in A$, of projections in $\mathfrak{B}$ will be said to be *increasing* (*decreasing*) if $\alpha \leq \beta$ implies $E_\alpha \leq E_\beta$ ($E_\beta \leq E_\alpha$).

([2]) The set clm $\{A_\alpha\}$ is the least closed linear manifold containing all the sets A_α.

Our first result shows that the weakest form of completeness for $\mathfrak{B}$ implies boundedness.

2.2. THEOREM. *If a B.A. of projections is σ-complete as an abstract B.A. then it is bounded.*

Proof. Suppose the B.A. $\mathfrak{B}$ is not bounded. We show first that $\mathfrak{B}$ contains a monotone decreasing sequence $\{E_n\}$ such that

$$|E_n| \geq n + |E_{n-1}|, \qquad\qquad n = 2, 3, \cdots.$$

We say a projection E has property (α) if $\sup_{F \leq E} |F| = \infty$. Clearly for any $E \in \mathfrak{B}$, either E or $I - E$ has property (α), and if E has (α), $F \leq E$, then either F or $E - F$ has (α). Let E_1 have (α). Then there is an $F_1 \leq E_1$ such that $|F_1| \geq 2 + 2|E_1|$. Let E_2 be the member of the pair $(F_1, E_1 - F_1)$ with (α). The inequality $|E_1 - F_1| \geq |F_1| - |E_1|$ shows $|E_2| \geq 2 + |E_1|$. An F_2 is now selected in E_2 such that $|F_2| \geq 3 + 2|E_2|$, etc. The construction proceeds by induction.

Now for each n let $G_n = E_n - E_{n+1}$. The projections G_n are disjoint and $\lim_{n \to \infty} |G_n| = \infty$. By selecting subsequences from the sequence $\{G_n\}$ we obtain a collection of mutually disjoint sequences of projections $\{H_{jk}\}$, $j = 1, 2, \cdots, k = 1, 2, \cdots$, such that

$$\lim_{k \to \infty} |H_{jk}| = \infty, \qquad\qquad j = 1, 2, \cdots,$$

Define $P_j = \bigvee_{k=1}^{\infty} H_{jk}$. Then $P_n P_m = 0$, $n \neq m$. The relation

$$\frac{|H_{mn}x|}{|x|} = \frac{|H_{mn}P_m x|}{|x|} \leq \frac{|P_m| \, |H_{mn}P_m x|}{|P_m x|}, \qquad P_m x \neq 0,$$

shows

$$|H_{mn}|_{P_m \mathfrak{X}} \geq \frac{|H_{mn}|}{|P_m|},$$

where the left side is the norm of H_{mn} as an operator in $P_m \mathfrak{X}$. Consequently,

$$\lim_{n \to \infty} |H_{mn}|_{P_m \mathfrak{X}} = \infty, \qquad\qquad m = 1, 2, \cdots.$$

Select a subsequence $\{n_i\}$ and unit vectors $x_i \in P_i X$ such that $|H_{in_i}x_i| > i$, $i = 1, 2, \cdots$. The projection $Q = \bigvee_{i=1}^{\infty} H_{in_i}$ cannot be bounded since

$$|Qx_i| = |QP_i x_i| = |H_{in_i}x_i| > i.$$

This contradiction completes the proof.

Theorem 2.2 has been proved by Lorch [14] under the assumptions that $\mathfrak{B}$ is complete and is generated by one-dimensional projections arising from a basis.

Hereafter if $\mathfrak{B}$ is σ-complete or complete, the symbol M will denote the bound for $\mathfrak{B}$.

2.3. LEMMA. *Let $\mathfrak{B}$ be a complete B.A. of projections in $\mathfrak{X}$ and let $\{E_\alpha\}$, $\alpha \in A$, be an increasing (decreasing) net in $\mathfrak{B}$. Then $\lim_\alpha E_\alpha x = (\bigvee_\alpha E_\alpha)x$ ($\lim_\alpha E_\alpha x = \bigwedge_\alpha E_\alpha x$), $x \in \mathfrak{X}$. If $\mathfrak{B}$ is σ-complete, corresponding results are true if the net $\{E_\alpha\}$ is a sequence $\{E_n\}$.*

Proof. Let $\{E_\alpha\}$ be increasing and $E_0 = \bigvee_\alpha E_\alpha$. Let $\epsilon > 0$ and $x \in X$. Since $E_0\mathfrak{X} = \text{clm} \{E_\alpha\mathfrak{X}\}$ we can find a vector $y = \sum_{i=1}^n \beta_i z_i$ and indices α_i such that $E_{\alpha_i} z_i = z_i$ and $|E_0 x - y| < \epsilon$. If $\alpha \geq \alpha_i$, $i = 1, \cdots, n$, then $E_\alpha y = y$. Consequently,

$$\big| E_\alpha x - E_0 x \big| \leq \big| E_\alpha x - y \big| + \big| y - E_0 x \big|$$
$$= \big| E_\alpha(E_0 x - y) \big| + \big| y - E_0 x \big| < (M + 1)\epsilon,$$

for $\alpha \geq \alpha_i$, showing $\lim_\alpha E_\alpha x = E_0 x$. The second conclusion follows from the formula $\bigwedge_\alpha E_\alpha = I - \bigvee_\alpha(I - E_\alpha)$ and what has just been proved. The proof for sequences is identical.

Given a B.A. of projections $\mathfrak{B}$ we denote by $\mathfrak{B}^*$ the B.A. of adjoints in $\mathfrak{X}^*$ of elements of $\mathfrak{B}$. When $\mathfrak{B}$ is complete we encounter a third form of completeness in $\mathfrak{B}^*$. By the $\mathfrak{X}$-topology in $\mathfrak{X}^*$ we shall mean the topology generated by neighborhoods of the form

$$N(\overset{*}{x_0}; x_1, \cdots, x_n, \epsilon) = \big\{ x^* \,\big|\, | x^*(x_i) - \overset{*}{x_0}(x_i) | < \epsilon, i = 1, \cdots, n \big\}.$$

2.4. DEFINITION. Let $\mathfrak{D}$ be a B.A. of projections in $\mathfrak{X}^*$ and let $F\mathfrak{X}^*$ be $\mathfrak{X}$-closed for every $F \in \mathfrak{D}$. Then $\mathfrak{D}$ is called $\mathfrak{X}$-*complete* ($\mathfrak{X}$-σ-*complete*) if for every subset (sequence) $\{F_\alpha\} \subseteq \mathfrak{D}$,

(a) $\mathfrak{X}^*$ admits the direct sum decomposition $\mathfrak{X}^* = \mathfrak{M} \oplus \mathfrak{N}$ where[3]

$$\mathfrak{M} = (\mathfrak{X}) - \text{clm} \{F_\alpha\mathfrak{X}^*\}, \qquad \mathfrak{N} = \bigcap_\alpha (I - F_\alpha)\mathfrak{X}^*,$$

(b) the projection F_0 with range $\mathfrak{M}$ and null manifold $\mathfrak{N}$ defined by this decomposition belongs to $\mathfrak{D}$.

It is easily verified that $\mathfrak{X}$-completeness implies completeness for $\mathfrak{D}$ as an abstract B.A., and that F_0 in the definition above is the projection $\bigvee_\alpha F_\alpha$. Moreover

$$\Big(\bigwedge_\alpha F_\alpha\Big) \mathfrak{X}^* = \bigcap_\alpha F_\alpha\mathfrak{X}^*, \quad \Big(I - \bigwedge_\alpha F_\alpha\Big) \mathfrak{X}^* = (\mathfrak{X}) - \text{clm} \big\{(I - F_\alpha)\mathfrak{X}^*\big\}.$$

2.5. LEMMA. *Let $\mathfrak{B}$ be a complete (σ-complete) B.A. of projections in $\mathfrak{X}$ and let B^* be the B.A. of adjoints in $\mathfrak{X}^*$ of elements of $\mathfrak{B}$. Then $\mathfrak{B}^*$ is $\mathfrak{X}$-complete ($\mathfrak{X}$-σ-complete) in $\mathfrak{X}^*$. In particular the manifolds $E^*\mathfrak{X}^*$, $E^* \in B^*$, are $\mathfrak{X}$-closed.*

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[3] The set $(\mathfrak{X}) - \text{clm} \{A_\alpha\}$ is the least $\mathfrak{X}$-closed linear manifold in $\mathfrak{X}^*$ containing all the sets A_α.

Proof. That $E^*\mathfrak{X}^* = \{x^* \mid (I-E)^*x^* = 0\}$ is $\mathfrak{X}$-closed is a standard property of the null manifold of an adjoint operator. Let $\{E_\alpha\}$ be an arbitrary set (sequence) of elements of $\mathfrak{B}$ and $E_0 = \bigvee E_\alpha$. Since we may replace $\{E_\alpha\}$ by the increasing net of its finite unions, we may suppose $\{E_\alpha\}$ is an increasing net. Then $E_0 x = \lim_\alpha E_\alpha x$, $x \in \mathfrak{X}$, by Lemma 2.3. Consequently $E_0^* x^*(x)$ $= x^* E_0 x = \lim x^* E_\alpha x = \lim E_\alpha^* x^* x$, $x \in \mathfrak{X}$. Thus $E_0^* \mathfrak{X}^* \subseteq (\mathfrak{X}) - \mathrm{clm}\ \{E_\alpha^* \mathfrak{X}^*\}$. But $E_0^* \mathfrak{X}^*$ is $\mathfrak{X}$-closed. Thus $E_0^* \mathfrak{X}^* = (\mathfrak{X}) - \mathrm{clm}\ \{E_\alpha^* \mathfrak{X}^*\}$. One shows similarly that $(I^* - E_0^*)\mathfrak{X}^* = \bigcap_\alpha (I^* - E_\alpha^*)\mathfrak{X}^*$.

We recall that any abstract Boolean algebra is isomorphic with the B.A. of all open and closed sets of a totally disconnected compact Hausdorff space called the *representation space* of the B.A. By a theorem of M. H. Stone [20] completeness of the B.A. is equivalent to the statement that the representation space is *extremely disconnected*; i.e. the closure of every open set is open. In this case corresponding to each Borel set e there is a unique open and closed set σ such that $(e - \sigma) \cup (\sigma - e)$ is of the first category. Moreover, each Borel function differs from a unique continuous function on a Borel set of the first category. If the B.A. is σ-complete, it is σ-isomorphic with the B.A. of Baire sets of its representation space modulo the σ-ideal of first category Baire sets. These facts are discussed in [5; 10; 20].

We shall denote the representation space of a B.A. of projections by Ω. It is important for what follows that we may regard a complete B.A. of projections as a projection valued measure defined on the Borel sets of Ω. The vector and scalar measures $E(\cdot)x$ and $x^*E(\cdot)x$, $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, are countably additive. If the B.A. of projections is only σ-complete we replace "Borel" by "Baire" in the foregoing. Similar remarks apply to the scalar measures $F(\cdot)x^*x$ if $\mathfrak{B}$ is an $\mathfrak{X}$-complete or $\mathfrak{X}$-σ-complete B.A. of projections in $\mathfrak{X}^*$.

A B.A. of projections is called *countably decomposable* if every collection of mutually disjoint elements of $\mathfrak{B}$ is at most countable. For each $x \in \mathfrak{X}$ we define $\mathfrak{M}(x) = \mathrm{clm}\ \{Ex \mid E \in \mathfrak{B}\}$. The subspace $\mathfrak{M}(x)$ is called the *cyclic subspace* generated by x. If $\mathfrak{B}$ is complete then for each $x \in \mathfrak{X}$ the projection $E_x = \bigwedge \{E \mid E \in \mathfrak{B}, Ex = x\}$ is called the *carrier projection* of x. The corresponding open-closed set in Ω, denoted by σ_x, is called the *carrier set* of x. Similar definitions apply if $\mathfrak{B}$ is an $\mathfrak{X}$-complete B.A. in $\mathfrak{X}^*$.

2.6. LEMMA. *Let $\mathfrak{B}$ be a σ-complete B.A. of projections in $\mathfrak{X}$. If $x \in \mathfrak{X}$, then $\mathfrak{M}(x)$ is invariant under $\mathfrak{B}$ and the restriction $\mathfrak{B}(x)$ of $\mathfrak{B}$ to $\mathfrak{M}(x)$ is a complete B.A. of projections in $\mathfrak{M}(x)$. Moreover, if $\mathfrak{B}$ is complete, the representation space $\Omega(x)$ of $\mathfrak{B}(x)$ may be identified with σ_x, the carrier set of x, in its relative topology.*

Proof. Fix $x \in \mathfrak{X}$ and let $\bar{E}$ denote the restriction of $E \in \mathfrak{B}$ to $\mathfrak{M}(x)$. Then $\mathfrak{B}(x)$ is clearly a bounded B.A. of projections in $\mathfrak{M}(x)$. Let $\{\bar{E}_n\} \subseteq \mathfrak{B}(x)$. Since we may replace $\{E_n\} \subseteq \mathfrak{B}$ by the sequence of its finite unions, there is no loss of generality in supposing $\{E_n\}$ is an increasing sequence in $\mathfrak{B}$. If $E_0 = \bigvee E_n$ in $\mathfrak{B}$, then $\lim_{n \to \infty} E_n y = E_0 y$, $y \in \mathfrak{M}(x)$ by Lemma 2.3. Thus $\bar{E}_0 = \bigvee \bar{E}_n$ in $\mathfrak{B}(x)$

and the conditions of Definition 2.1 are satisfied. Consequently $\mathfrak{B}(x)$ is σ-complete in $\mathfrak{M}(x)$.

Now $\mathfrak{B}(x)$ determines the countably additive vector valued measure $\overline{E}(\cdot)x$ on the Baire sets of $\Omega(x)$. By a result of Bartle, Dunford, and Schwartz [2], every countably additive vector valued measure is absolutely continuous with respect to a finite positive measure defined on the same σ-field. Consequently, if $\{\sigma_\alpha\}$ is a family of disjoint open and closed sets in $\Omega(x)$, $\overline{E}(\sigma_\alpha)x=0$ for all but countably many values of α. By the definition of $\mathfrak{M}(x)$ this implies $\overline{E}(\sigma_\alpha)=0$ except for a countable set of α. Thus $\mathfrak{B}(x)$ is countably decomposable. It is known that a σ-complete countably decomposable B.A. is complete in the sense that every set has a least upper bound which is the least upper bound of a countable subset of the set[4]. Thus since $\mathfrak{B}(x)$ is σ-complete in the sense of Definition 2.1, it is complete in the sense of Definition 2.1. The proof of the final statement is left to the reader.

The next two results show that any σ-complete B.A. of projections may be imbedded in a smallest complete B.A. of projections.

2.7. THEOREM. *If $\mathfrak{B}$ is a σ-complete B.A. of projections, then $\overline{\mathfrak{B}}^s$ is complete.*

Proof. It is easily seen that $\overline{\mathfrak{B}}^s$ is a bounded B.A. of projections. Let $\{E_\alpha\}$, $\alpha \in A$, be a family of projections in $\mathfrak{B}$. Since we may replace $\{E_\alpha\}$ by the net of its finite unions we may suppose it is an increasing net. If $x \in X$, then $\lim_\alpha E_\alpha x = \lim_\alpha \overline{E}_\alpha x$ exists since $\mathfrak{B}(x)$ is complete (Lemmas 2.3 and 2.6). Clearly the strong limit E_0 of the E_α is the projection $\bigvee_\alpha E_\alpha$ of Definition 2.1. Similarly we construct the infimum of an arbitrary set in $\mathfrak{B}$.

Now let $\mathfrak{B}_1$ be the B.A. generated by arbitrary sups and infs of projections in $\mathfrak{B}$. Since $\mathfrak{B}_1(x)=\mathfrak{B}(x)$, $x \in \mathfrak{X}$, we may construct arbitrary sups and infs of elements of $\mathfrak{B}_1$. By transfinite induction we obtain a monotone family $\{\mathfrak{B}_\alpha\}$ of B.A.'s, indexed on the ordinals α less than some large ordinal η, such that each $\mathfrak{B}_\alpha$ contains arbitrary sups and infs of the $\mathfrak{B}_\beta$ for $\beta < \alpha$. If α_0 is the least ordinal for which $\mathfrak{B}_{\alpha_0} = \bigcup_{\alpha < \alpha_0} \mathfrak{B}_\alpha$, then $\mathfrak{B}_{\alpha_0}$ is complete. Clearly $\mathfrak{B}_\alpha \subseteq \overline{\mathfrak{B}}^s$. The conclusion will follow if we can show that a complete B.A. of projections is strongly closed. However, this follows from the next result:

2.8. THEOREM. *A complete B.A. of projections contains every projection in the weakly closed algebra which it generates.*

This was proved in [1, p. 405], under the assumption that $\mathfrak{X}$ is reflexive and the B.A. is bounded. The boundedness assumption is unnecessary in view of Theorem 2.2. The proof in [1] is valid in any Banach space if where

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[4] The completeness is proved by von Neumann, *Lectures on continuous geometry* III, Princeton, 1936. The second conclusion is evident from the following proof for which we are indebted to Alfred Horn: If $\mathfrak{A} \subseteq \mathfrak{B}$ let $\mathfrak{A}_1$ denote all suprema of countable subsets of $\mathfrak{A}$. Then $\mathfrak{A}_1$ contains a maximal subset well ordered in the ordering of $\mathfrak{B}$. As $\mathfrak{B}$ is countably decomposable, this chain is countable and its supremum has the required properties.

Lemma 3.1 and Theorem 1.1 of [1] are used one substitutes Lemma 2.3 above and Theorem 17 of [7][(5)].

2.9. LEMMA. *If $\mathfrak{X}$ is weakly complete, any bounded B.A. $\mathfrak{B}$ of projections may be imbedded in a σ-complete B.A. of projections contained in $\overline{\mathfrak{B}}^{s}$.*

Lemma 2.9 is due to Dunford. The proof given in [5, p. 578], for a special case, is valid in general. The fact that if $\mathfrak{B}$ is bounded, then $\overline{\mathfrak{B}}^{s}$ is complete in a reflexive space was observed by Dunford in [6]. A different proof may be found in [1]. The referee has observed that this special case of Theorem 2.7 also follows from an earlier result of Day [3, Theorem 6].

2.10. COROLLARY. *If $\mathfrak{B}$ is bounded in a weakly complete space, then $\overline{\mathfrak{B}}^{s}$ is complete.*

3. **Positive functionals.** We have made use earlier of the theorem of [2] that a countably additive vector measure is dominated by a finite positive measure. We shall show in this section that when the vector measure is determined by a σ-complete B.A. of projections, i.e. has the form $E(\cdot)x_0$, $x_0 \in \mathfrak{X}$, $E \in \mathfrak{B}$, the dominating scalar measure may be chosen to have the form $x_0^* E(\cdot)x_0$ where $x_0^* \in \mathfrak{X}^*$. The proof of the existence of the scalar measure in this case is different from that in [2]. We remark that if $\mathfrak{B}$ is a B.A. of self-adjoint projections in Hilbert space one may take for x_0^* the functional determined by x_0 itself, i.e., $x_0^* E(\cdot)x_0 = (E(\cdot)x_0, x_0)$.

3.1. THEOREM. *Let $\mathfrak{B}$ be a σ-complete B.A. of projections in $\mathfrak{X}$. Then given $x_0 \in \mathfrak{X}$ there is a linear functional $x_0^* \in \mathfrak{X}^*$ with the properties*
   (i) $x_0^* E x_0 \geqq 0$, $E \in \mathfrak{B}$.
   (ii) *If for any $E \in \mathfrak{B}$, $x_0^* E x_0 = 0$, then $E x_0 = 0$.*

Proof. Since we may replace $\mathfrak{B}$ by $\overline{\mathfrak{B}}^{s}$, we may assume without loss of generality that $\mathfrak{B}$ is complete (cf. Theorem 2.7). Also in view of Lemma 2.6 and the Hahn-Banach theorem, there is no loss in generality in supposing $\mathfrak{X} = \text{clm} \{Ex_0 | E \in \mathfrak{B}\} = \mathfrak{M}(x_0)$. Since $\mathfrak{B}^*$ is $\mathfrak{X}$-complete in $\mathfrak{X}^*$ by Lemma 2.5, there is associated with each $y^* \in \mathfrak{X}^*$ a unique *carrier* projection

$$E_{y^*}^* = \bigwedge \{E^* \mid E^* y^* = y^*\}.$$

Let $\mathfrak{F}$ be a family of projections in $\mathfrak{B}^*$ maximal with respect to the properties: (a) the members of $\mathfrak{F}$ are disjoint, and (b) each projection in $\mathfrak{F}$ is a carrier projection. The assumption that $\mathfrak{X} = \text{clm} \{Ex_0 | E \in \mathfrak{B}\}$ implies $\mathfrak{B}$ is countably decomposable (cf. the proof of Lemma 2.6). Since the property of being countably decomposable is clearly inherited by $\mathfrak{B}^*$ from $\mathfrak{B}$, the family $\mathfrak{F}$ is at most countable. Letting $\mathfrak{F} = \{E_{y_n^*}^*\}$ where $|y_n^*| = 1$, define $y_0^* = \sum_{n=1}^{\infty} y_n^*/2^n$.

[(5)] The inequality of [1, p. 406, line 18] requires an additional term $|E_{n\epsilon}A_n y - A_n y|$ on the right-hand side. This dictates minor corrections in the inequalities of lines 19 and 21 which, however, do not affect the validity of the argument.

We assert that the carrier of y_0^* is I^*, the identity in $\mathfrak{X}^*$. For let $E_0^* = I^* - E_{y_0^*}^*$. Then $E_0^* y_n^* = 2^n E_{y_n^*} E_0^* y_0^* = 0$ for each n, and thus $E_{y_n}^* \leq (I^* - E_0^*)$, i.e. $E_0^* E_{y_n}^* = 0$ for all n. If E_0^* is not zero it contains a nonzero carrier projection, contradicting the maximality of $\mathfrak{F}$[(6)].

We now obtain the functional x_0^* from y_0^*. Regarding $\mathfrak{B}$ as a spectral measure on the Borel sets Σ of Ω we define the positive measures

$$\mu_1(\sigma) = \text{tot. var.}_\sigma \, \Re y_0^* E(\cdot) x_0, \qquad\qquad \sigma \in \Sigma,$$

$$\mu_2(\sigma) = \text{tot. var.}_\sigma \, \Im y_0^* E(\cdot) x_0, \qquad\qquad \sigma \in \Sigma,$$

and let $\mu = \mu_1 + \mu_2$. Let $\mathfrak{X}_0$ be the set of all vectors of the form $\sum_{i=1}^n \alpha_i E(\sigma_i) x_0$, where the sets σ_i are disjoint. Then $\mathfrak{X}_0$ is dense in $\mathfrak{X}$. Let the functional θ on $\mathfrak{X}_0$ be defined by

$$\theta\left(\sum_{i=1}^n \alpha_i E(\sigma_i) x_0 \right) = \sum_{i=1}^n \alpha_i \mu(\sigma_i).$$

It will be shown that θ is continuous on $\mathfrak{X}_0$. Let $\Omega = e_+ \cup e_- = f_+ \cup f_-$ be two Hahn decompositions of Ω into disjoint Borel sets such that

$$\mu_1(\sigma) = \Re y_0^* E(\sigma \cap e_+) x_0 - \Re y_0^* E(\sigma \cap e_-) x_0, \qquad\qquad \sigma \in \Sigma,$$

$$\mu_2(\sigma) = \Im y_0^* E(\sigma \cap f_+) x_0 - \Im y_0^* E(\sigma \cap f_-) x_0, \qquad\qquad \sigma \in \Sigma.$$

Given $z = \sum_{i=1}^n \alpha_i E(\sigma_i) x_0$ we define the operators

$$A = \sum_{i=1}^n \epsilon_{i1} E(\sigma_i \cap e_+) + \sum_{i=1}^n \epsilon_{i2} E(\sigma_i \cap e_-),$$

$$B = \sum_{i=1}^n \epsilon_{i3} E(\sigma_i \cap f_+) + \sum_{i=1}^n \epsilon_{i4} E(\sigma_i \cap f_-),$$

where the quantities ϵ_{ij} are defined to be

$$\epsilon_{i1} = \frac{\Re y_0^* E(\sigma_i \cap e_+) x_0}{y_0^* E(\sigma_i \cap e_+) x_0}, \qquad \epsilon_{i2} = -\frac{\Re y_0^* E(\sigma_i \cap e_-) x_0}{y_0^* E(\sigma_i \cap e_-) x_0},$$

$$\epsilon_{i3} = \frac{\Im y_0^* E(\sigma_i \cap f_+) x_0}{y_0^* E(\sigma_i \cap f_+) x_0}, \qquad \epsilon_{i4} = -\frac{\Im y_0^* E(\sigma_i \cap f_-) x_0}{y_0^* E(\sigma_i \cap f_-) x_0},$$

if the denominators are not zero, and to be zero otherwise. Then $\left| \epsilon_{ij} \right| \leq 1$ and

$$\theta(z) = \sum_{i=1}^n \alpha_i \mu(\sigma_i) = \sum_{i=1}^n \alpha_i y_0^* E(\sigma_i)(A + B) x_0$$

$$= (A^* + B^*) y_0^* \left(\sum_{i=1}^n \alpha_i E(\sigma_i) x_0 \right) = (A^* + B^*) y_0^* z.$$

[(6)] This argument is familiar in the theory of W^* algebras.

But $|A^*|$, $|B^*| \leqq 4M$ by [7, Lemma 6]. Thus $|\theta(z)| \leqq 8M|y_0^*|\,|z|$. The extension x_0^* of θ to $\mathfrak{X}$ has the property that $x_0^*E(\sigma)x_0 \geqq 0$, $\sigma \in \Sigma$. Consequently if $x_0^*E(\sigma)x_0 = 0$ for some set σ, $x_0^*E(\delta)x_0 = 0$ for $\delta \subseteq \sigma$, which implies

$$\text{tot. var. } {}_\sigma y_0^* E(\cdot)x_0 = 0.$$

Thus $E^*(\sigma)y_0^*x = 0$ for each $x \in \mathfrak{X}_0$, and consequently $E^*(\sigma)y_0^* = 0$. Since the carrier of y_0^* is I^*, $E^*(\sigma) = 0$, i.e. $E(\sigma) = 0$. This completes the proof.

3.2. THEOREM. *The weak and strong operator topologies coincide on a complete B.A. of projections*(7).

Proof. The result follows easily from Theorem 3.1 and a theorem of Pettis [17] (for a proof of his theorem see [12]). However, we shall give another proof not using the result of Pettis. Let $\{E_\alpha\}$, $\alpha \in A$, be a net in $\mathfrak{B}$ such that $\lim_\alpha x^*E_\alpha x = x^*E_0 x$, $x^* \in \mathfrak{X}^*$, $x \in \mathfrak{X}$, $E_0^2 = E_0$ (and thus $E_0 \in \mathfrak{B}$ by Theorem 2.8), and suppose that $\lim E_\alpha x_0 \neq E_0 x_0$. There is no loss in generality in supposing $E_0 x_0 = 0$. Then there is a constant $\gamma > 0$ and a cofinal subnet $\{E_\beta\}$, $\beta \in B \subseteq A$, such that

(1) $|E_\beta x_0| \geqq \gamma,$ $\beta \in B.$

Let $E_\beta = E(\sigma_\beta)$, $\sigma_\beta \in \Sigma$. Then $x_0^*E(\sigma_\beta)x_0 = \mu(\sigma_\beta) \to 0$ where x_0^* is the functional of Theorem 3.1. We select a sequence $e_n = \sigma_{\beta_n}$ such that $\mu(e_n) \to 0$. The characteristic functions k_{e_n} converge to zero in μ-measure, and consequently a subsequence k_{δ_n} converges to zero except possibly on a set δ_0 for which $\mu(\delta_0) = 0$. By Theorem 3.1, $\mu(\delta_0) = 0$ implies $E(\delta_0)x_0 = 0$. Let

$$\theta_n = \{\omega \mid \omega \in \Omega, \; k_{\delta_i}(\omega) = 0, \; i \geqq n\}.$$

Then $\theta_1 \subseteq \theta_2 \subseteq \cdots$ and $E(\Omega - \bigcup_{i=1}^n \theta_i)x_0 \to E(\delta_0)x_0 = 0$. Consequently by Lemma 2.3, $\lim_{n\to\infty} E(\theta_n)x_0 = x_0$. Since

$$|E(\delta_m)x_0| = |E(\delta_m)E(\Omega - \theta_m)x_0|$$
$$\leqq M|E(\Omega - \theta_m)x_0|,$$

the inequality (1) is contradicted for large values of m. This contradiction completes the proof.

In view of Corollary 2.10 we have

3.3. COROLLARY. *The strong and weak operator topologies coincide on any bounded B.A. of projections in a weakly complete space.*

Let $\mathfrak{X}$ be a Hilbert space. The identity

$$|(E_\alpha - E_0)x|^2 = (E_\alpha x, x) + (E_0 x, x) - (E_\alpha x, E_0 x) - (E_0 x, E_\alpha x)$$

satisfied by (not necessarily commuting) self-adjoint projections shows Corol-

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(7) This result answers affirmatively a question raised in [1, footnote p. 405].

lary 3.3 is immediate in the case of self-adjoint projections. If the projections in $\mathfrak{B}$ are not self-adjoint, Corollary 3.3 follows from a theorem of Mackey [15, p. 146]: *There exists a bicontinuous operator A such that $P = AEA^{-1}$ is self-adjoint for every $E \in \mathfrak{B}$.* (This was proved in a special case by Lorch [14]. See also Wermer [21].) The referee has observed that the family $\{U\} = \{I - 2E \mid E \in \mathfrak{B}\}$ forms a bounded commutative group, and thus Mackey's theorem follows from Nagy's theorem [16] (see also Day [4, Theorem 8] (that every bounded commutative group of operators in Hilbert space is equivalent to a unitary group.

We now show by an example (due to R. R. Christian) that the conclusion of Theorem 3.2 need not hold if $\mathfrak{B}$ is not σ-complete. Let $\mathfrak{X}$ be the space (m) of bounded sequences. For each subset σ of positive integers let $E(\sigma)$ be the projection mapping $y = [\xi_n] \in (m)$ into $[\delta_\sigma^n \xi_n]$ where $\delta_\sigma^n = 1$ if $n \in \sigma$, and zero otherwise. Then $|E(\sigma)| = 1$. Let $\mathfrak{B}$ be the B.A. of such projections. Then $\mathfrak{B}$ is $\mathfrak{Y}$-complete as the adjoint of the corresponding B.A. in $\mathfrak{Y} = l_1$ ($\mathfrak{Y}^* = (m)$), but $\mathfrak{B}$ is not σ-complete. It is known that $\mathfrak{X}^* = (ba)$, the space of finitely additive measures μ on the positive integers such that

$$|\mu| = \sup_\pi \sum_{\sigma_i \in \pi} |\mu(\sigma_i)|,$$

where π is any finite partition of the integers. If $\mu \in (ba)$, $\mu(\{n\}) \to 0$, and thus for each $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, $x^* E(\{n\}) x = \xi_i \mu(\{n\}) \to 0$, where $\{n\}$ is the set consisting of the integer n. Consequently $\lim_{n \to \infty} E(\{n\}) = 0$ weakly. However the sequence does not converge to zero strongly.

It is interesting to note also that a sequence of projections may converge weakly to an operator which is not a projection. By a result of Dye [9, Lemma 2.3], if $\mathfrak{B}$ is a complete nonatomic B.A. of self-adjoint projections in Hilbert space, the weak closure of $\mathfrak{B}$ fills out the positive part of the unit sphere in the algebra generated by $\mathfrak{B}$. His argument may be extended to the case of a complete nonatomic B.A. in any reflexive space using the results of [1].

4. **Operator algebras generated by projections.** If $\mathfrak{B}$ is a bounded B.A. of projections, we denote by $\mathfrak{A}(\mathfrak{B})$ and $\mathfrak{W}(\mathfrak{B})$ the algebras generated by $\mathfrak{B}$ in the uniform and weak operator topologies respectively. By a theorem of Dunford [7; p. 348], $\mathfrak{A}(\mathfrak{B})$ is equivalent to the algebra $C(\mathfrak{M})$ of continuous functions on its space $\mathfrak{M}$ of maximal ideals. One easily identifies $\mathfrak{M}$ with the representation space Ω of $\mathfrak{B}$. If $\mathfrak{B}$ is σ-complete this equivalence is given explicitly by the correspondence $h \to \int_\Omega h(\omega) E(d\omega)$, $h \in C(\Omega)$. In this section we shall give further characterizations of $\mathfrak{A}(\mathfrak{B})$ and of $\mathfrak{W}(\mathfrak{B})$ more closely related to the underlying space $\mathfrak{X}$. An application is made to the theory of spectral operators of scalar type (cf. [7]).

We shall say that a B.A. of projections $\mathfrak{B}$ has *simple spectrum* if for some $x_0 \in \mathfrak{X}$, $\mathfrak{X} = \mathfrak{M}(x_0) = \text{clm} \{Ex_0 \mid E \in \mathfrak{B}\}$. The proof of the next lemma is due to J. Schwartz.

4.1. LEMMA. *Let $\mathfrak{B}$ be a complete B.A. of projections with simple spectrum, i.e. $\mathfrak{X} = \mathfrak{M}(x_0)$, $x_0 \in \mathfrak{X}$. Let x_0^* be the functional associated with x_0 by Theorem 3.1. Then*

$$\mathfrak{X}^* = (\mathfrak{X}) - \mathrm{clm}\{E^* x_0^* \mid E^* \in \mathfrak{B}^*\}.$$

Proof. Let $\mathfrak{K} = \mathrm{sp}\ \{E^* x_0^* \mid E^* \in B^*\}$. To show K is X-dense in X^* it is sufficient to show that if $x \in \mathfrak{X}$ and $y^* x = 0$, $y^* \in \mathfrak{K}$, then $x = 0$; i.e. if the measure $E^*(\sigma) x_0^* x = x_0^* E(\sigma) x \equiv 0$, $\sigma \in \Sigma$, then $x = 0$. Since $\mathfrak{X} = \mathfrak{M}(x_0)$ let $\{f_n\}$ be a sequence of finite linear combinations of characteristic functions such that

$$x = \lim_{n \to \infty} \int_\Omega f_n(\omega) E(d\omega) x_0.$$

Then for each $\sigma \in \Sigma$,

$$\lim_{n \to \infty} \int_\sigma f_n(\omega) x_0^* E(d\omega) x_0 = x_0^* E(\sigma) x = 0$$

uniformly in σ. Since

$$\int_\Omega |f_n(\omega)|\ x_0^* E(d\omega) x_0 \leq 4 \sup_{\sigma \in \Sigma} \left| \int_\sigma f_n(\omega) x_0^* E(d\omega) x_0 \right|,$$

we see $f_n \to 0$ in $L_1(\Omega, \Sigma, x_0^* E(\cdot) x_0)$. Thus a subsequence g_n of f_n converges to zero almost everywhere and almost uniformly. Now let $\{\delta_m\}$ be a decreasing sequence of Borel sets such that $x_0^* E(\delta_m) x_0 \to 0$ and $g_n \to 0$ uniformly on δ_m', $m = 1, 2, \cdots$. For each m

$$x = \lim_{n \to \infty} \int_{\delta_m} g_n(\omega) E(d\omega) x_0 + \lim_{n \to \infty} \int_{\Omega - \delta_m} g_n(\omega) E(d\omega) x_0$$

$$= \lim_{n \to \infty} E(\delta_m) \int_\Omega g_n(\omega) E(d\omega) x_0 = E(\delta_m) x_0.$$

Therefore $x = E(\bigcap_{m=1}^\infty \delta_m) x$. Since $x_0^* E(\bigcap_{m=1}^\infty \delta_m) x_0 = 0$, $E(\bigcap_{m=1}^\infty \delta_m) = 0$ and thus $x = 0$.

The next result has been proved by F. Wolf [22] in the case that X is a Hilbert space.

4.2. THEOREM. *Let $\mathfrak{B}$ be a complete B.A. of projections with simple spectrum. Then the algebra $\mathfrak{A}(\mathfrak{B})$ generated by $\mathfrak{B}$ in the uniform topology consists of all bounded operators commuting with $\mathfrak{B}$.*

Proof. Let $\mathfrak{X} = \mathfrak{M}(x_0)$ and select x_0^* with the properties of Theorem 3.1. Then if A commutes with $\mathfrak{B}$, the set function $x_0^* A E(\cdot) x_0$ is absolutely continuous with respect to $x_0^* E(\cdot) x_0$ and by the Radon-Nikodym theorem there is a Borel measurable function $h(\cdot)$, integrable with respect to $x_0^* E(\cdot) x_0$, such that

$$x_0^* A E(\sigma) x_0 = \int_\sigma h(\omega) x_0^* E(d\omega) x_0, \qquad\qquad \sigma \in \Sigma.$$

Let $\sigma_n = \{\omega \mid |h(\omega)| \le n\}$, and $A_n = \int_{\sigma_n} h(\omega) E(d\omega)$. Then for each n

$$x_0^* A E(\sigma_n) E(\sigma) x_0 = \int_{\sigma \cap \sigma_n} h(\omega) E(d\omega) x_0$$
$$= x_0^* A_n E(\sigma) x_0, \qquad\qquad \sigma \in \Sigma.$$

Hence $x_0^* A E(\sigma_n) x = x_0^* A_n x$, $x \in X$, and $E^*(\sigma_n) A^* x_0^* = A_n^* x_0$. Using commutativity,

$$A^* E^*(\sigma_n) E^*(\sigma) x_0^* = A_n^* E^*(\sigma) x_0^*, \qquad\qquad \sigma \in \Sigma.$$

Since clm $\{E^* x_0^* \mid E^* \in \mathfrak{B}^*\}$ is $\mathfrak{X}$-dense in $\mathfrak{X}^*$ by Lemma 4.1, $A^* E^*(\sigma_n) = A_n^*$, and hence $E(\sigma_n) A = A_n$. Thus the operators A_n are uniformly bounded. It follows that $h(\cdot)$ is essentially bounded on Ω and

$$A x = \lim_{n \to \infty} E(\sigma_n) A x = \int_\Omega h(\omega) E(d\omega) x, \qquad\qquad x \in \mathfrak{X}.$$

Since Ω is extremely disconnected we may suppose $h(\cdot)$ is continuous. The result now follows from the fact, noted earlier, that

$$\mathfrak{A}(\mathfrak{B}) = \left\{ A \mid A = \int_\Omega h(\omega) E(d\omega), \ h \in C(\Omega) \right\}.$$

4.3. Theorem. *Let $\mathfrak{B}$ be a complete B.A. of projections. Then the algebra $\mathfrak{A}(\mathfrak{B})$ generated by $\mathfrak{B}$ in the uniform operator topology is the family of all operators which leave invariant every linear manifold invariant under $\mathfrak{B}$.*

Proof. Each operator in $\mathfrak{A}(\mathfrak{B})$ clearly has the required invariance property. Conversely, suppose that the bounded operator A leaves invariant each manifold invariant under $\mathfrak{B}$. Equivalently, $A\mathfrak{M}(x) \subseteq \mathfrak{M}(x)$, $x \in \mathfrak{X}$. We shall show as in the previous proof that $A = \int_\Omega h(\omega) E(d\omega)$ where $h \in C(\Omega)$. An application of Zorn's Lemma shows $\mathfrak{B}$ contains a family of disjoint carrier projections $\{E_\alpha\}$, such that $\vee_\alpha E_\alpha = I$. The corresponding open-closed sets σ_α are disjoint and $\cup_\alpha \sigma_\alpha$ is dense in Ω. If for each α we can produce a function p_α continuous on σ_α such that $A E_\alpha = \int_{\sigma_\alpha} p_\alpha(\omega) E(d\omega)$, the function defined by $p(\omega) = p_\alpha(\omega)$, $\omega \in \sigma_\alpha$, $p(\omega) = 0$, $\omega \in \Omega - \cup_\alpha \sigma_\alpha$, is a Borel function, and thus differs from a continuous function h on a set of the first category. It follows that it is sufficient to prove the sufficiency under the assumption that there is an $x_0 \in \mathfrak{X}$ whose carrier projection is I.

We next observe that the condition $A\mathfrak{M}(x) \subseteq \mathfrak{M}(x)$, $x \in \mathfrak{X}$, implies

$$E(\sigma) A x = E(\sigma) A E(\sigma) x + E(\sigma) A E(\sigma') x$$
$$= A E(\sigma) x, \qquad\qquad \sigma \in \Sigma, \ x \in \mathfrak{X},$$

i.e. A commutes with $\mathfrak{B}$. Since the restriction of $\mathfrak{B}$ to $\mathfrak{M}(x_0)$ has simple spectrum, it follows from Theorem 4.2 and Lemma 2.6 that there is a continuous function h such that

$$Ax = \int_\Omega h(\omega)E(d\omega)x, \qquad\qquad x \in \mathfrak{M}(x_0).$$

It will be shown this equation holds for all $x \in \mathfrak{X}$. If $y_0 \in \mathfrak{X}$, by the same argument there is a function $q(\cdot)$ continuous on σ_{y_0}, the carrier set of y_0, such that

$$Ay = \int_{\sigma_{y_0}} q(\omega)E(d\omega)y, \qquad\qquad y \in \mathfrak{M}(y_0).$$

If $q(\omega) \neq h(\omega)$ for some $\omega \in \sigma_{y_0}$, there is an open-closed set $\sigma_0 \subseteq \sigma_{y_0}$ and a real number $\cdot \gamma > 0$ such that $|h(\omega) - q(\omega)| > \gamma$, $\omega \in \sigma_0$. Since we may replace y_0 by $E(\sigma_0)y_0$, there is no loss of generality in supposing $E(\sigma_0)y_0 = y_0$ and that q is defined on all Ω, equaling zero on σ'_{y_0}. Then clearly $\mathfrak{M}(x_0) \cap \mathfrak{M}(y_0) = (0)$.

By hypothesis $A\mathfrak{M}(x_0+y_0) \subseteq \mathfrak{M}(x_0+y_0)$. Thus

$$w_0 = A(x_0 + y_0) - \int_\Omega h(\omega)E(d\omega)(x_0 + y_0)$$

$$= \int_\Omega (q(\omega) - h(\omega))E(d\omega)y_0 \in \mathfrak{M}(x_0 + y_0).$$

Thus $y_0 = \int_{\sigma_{y_0}} [q(\omega) - h(\omega)]^{-1}E(d\omega)w_0 \in \mathfrak{M}(x_0+y_0)$, and hence $x_0 \in \mathfrak{M}(x_0+y_0)$.

For convenience write $z_0 = x_0+y_0$. By Theorem 3.1 there is a functional z_0^* such that $z_0^*E(\sigma)z_0 \geq 0$, $\sigma \in \Sigma$, with the property that if $z_0^*E(\delta)z_0 = 0$ for some $\delta \in \Sigma$, then $E(\delta)z_0 = 0$. However, if $E(\delta)z_0 = 0$, the fact $\mathfrak{M}(x_0) \cap \mathfrak{M}(y_0) = (0)$ implies $E(\delta)y_0 = 0$. Thus $z_0^*E(\cdot)y_0$ is absolutely continuous with respect to $z_0^*E(\cdot)z_0$, and by the Radon-Nikodym theorem there is a function $r \in L_1(z_0^*E(\cdot)z_0)$ such that

$$z_0^*E(\sigma)y_0 = \int_\sigma r(\omega)z_0^*E(d\omega)z_0, \qquad\qquad \sigma \in \Sigma.$$

Since Ω is extremely disconnected we may suppose that r is continuous. Select an open-closed set δ and constant $k > 0$ such that $k^{-1} < |r(\omega)| < k$, $\omega \in \delta$. Then $E(\delta)y_0 \neq 0$. If $D_\delta = \int_\delta r(\omega)E(d\omega)$,

$$E^*(\sigma)z_0^*E(\delta)y_0 = E^*(\sigma)z_0^*D_\delta z_0, \qquad\qquad \sigma \in \Sigma.$$

It follows from Lemma 4.1 (applied in the space $\mathfrak{M}(z_0)$) that

$$E(\delta)y_0 = D_\delta z_0 = D_\delta x_0 + D_\delta y_0.$$

Since $\mathfrak{M}(x_0) \cap \mathfrak{M}(y_0) = (0)$, $D_\delta x_0 = 0$. Thus $\int_\sigma r(\omega)x_0^*E(d\omega)x_0 = 0$, $\sigma \subseteq \delta$, where

x_0^* is associated with x_0 as in Theorem 3.1. This implies $r(\omega) = 0$, $\omega \in \delta$. This contradiction completes the proof.

Observe that if in Theorem 4.3 we take for $\mathfrak{B}$ the resolution of the identity of a bounded self-adjoint operator T in a separable Hilbert space, then for each $x \in \mathfrak{X}$, the manifold $\mathfrak{M}(x)$ admits a self-adjoint projection P commuting with $\mathfrak{B}$. Thus the condition $A\mathfrak{M}(x) \subseteq \mathfrak{M}(x)$ is satisfied by every operator A in the second commutant of T, since $Ax = APx = PAx$. Since $\mathfrak{A}(E(\cdot, T))$ consists of all Borel functions of T, we obtain a well known result of von Neumann (first formulated explicitly by F. Riesz [18]).

4.4. COROLLARY. *If T is a bounded self-adjoint operator in a separable Hilbert space, every operator A which commutes with every operator commuting with T is a Borel function of T.*

4.5. THEOREM. *Let $\mathfrak{B}$ be a σ-complete B.A. of projections in a Banach space. The following statements are equivalent:*
  (a) *$\mathfrak{B}$ is complete.*
  (b) *$\mathfrak{B}$ is strongly closed.*
  (c) *$\mathfrak{A}(\mathfrak{B}) = \mathfrak{W}(\mathfrak{B})$, i.e. $\mathfrak{A}(\mathfrak{B})$ is weakly closed.*
If $\mathfrak{X}$ is weakly complete the hypothesis that $\mathfrak{B}$ is σ-complete may be replaced by the hypothesis that $\mathfrak{B}$ is bounded.

Proof. The equivalence of (a) and (b) follows from Theorems 2.7, 2.8, and Lemma 2.9. Now let $\mathfrak{B}$ be complete. It is known that the weak closure of a convex set of bounded operators coincides with the strong closure (cf. [1, p. 404]). Thus an operator A in the weak closure of $\mathfrak{A}(\mathfrak{B})$ is a strong limit of operators in $\mathfrak{A}(\mathfrak{B})$. Thus $A\mathfrak{M}(x) \subseteq \mathfrak{M}(x)$, $x \in \mathfrak{X}$, and by Theorem 4.3, $A \in \mathfrak{A}(\mathfrak{B})$. Finally let $\mathfrak{A}(\mathfrak{B})$ be weakly closed. If $E \in \overline{\mathfrak{B}}$ there is an open and closed set σ in Ω, the representation space of $\mathfrak{B}$, such that $E = E(\sigma)$. But then $E \in \mathfrak{B}$, and thus (c) implies (b).

We recall that a bounded operator S in a complex Banach space $\mathfrak{X}$ is a scalar type spectral operator of class $\mathfrak{X}^*$ if $S = \int_{\sigma(S)} \lambda E(d\lambda)$, where $E(\cdot)$ is a projection valued measure defined on the Borel sets of the spectrum $\sigma(S)$ of S such that $x^*E(\cdot)x$, is countably additive for all $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$. It is known that this last condition implies the countable additivity of the vector valued set functions $E(\cdot)x$, $x \in \mathfrak{X}$. If $\mathfrak{B}$ is the range of the resolution of the identity of S, then $\mathfrak{B}$ is a σ-complete B.A. Consequently $\overline{\mathfrak{B}}$ is complete by Theorem 2.7, and the weakly closed algebra generated by S is a subset of the weakly closed algebra $\mathfrak{A}(\overline{\mathfrak{B}})$. Each operator in $\mathfrak{A}(\overline{\mathfrak{B}})$ is a scalar type spectral operator of class $\mathfrak{X}^*$ since it is of the form $\int_{\overline{\Omega}} h(\omega) \overline{E}(d\omega)$, where $\overline{\Omega}$ is the representation space of $\overline{\mathfrak{B}}$ (cf. [7, p. 341]). Thus we have proved:

4.6. THEOREM. *If S is a scalar type spectral operator of class $\mathfrak{X}^*$, every operator in the weakly closed algebra generated by S is a scalar type spectral operator of class $\mathfrak{X}^*$.*

This result was proved in [1] for the special case of a reflexive space by a different method. There is now no difficulty in adapting the proof of Theorem 4.8 of [1] (cf. [1, p. 410] for definitions) to obtain:

4.7. THEOREM. *The weakly closed algebra generated by a scalar type spectral operator A of class X* with real spectrum consists of all extended bounded Baire functions of A.*

It would be interesting to know whether if $\mathfrak{B}$ is complete $\mathfrak{A}(\mathfrak{B})$ equals its second commutant. In view of Theorem 4.3 it would be sufficient to show that for each x there is an element Q in the commutant of $\mathfrak{A}(\mathfrak{B})$ with $Q\mathfrak{X}$ dense in $\mathfrak{M}(x)$. We have been able to establish this fact only in the case $\mathfrak{B}$ is atomic. A positive answer would yield a corresponding generalization of Corollary 4.4. A related question is whether each of the manifolds $\mathfrak{M}(x)$ admits a bounded projection commuting with $\mathfrak{B}$. This need not be the case if $\mathfrak{B}$ is not σ-complete. For let $\mathfrak{X} = (m)$ and $\mathfrak{B}$ be defined as in the example following Corollary 3.3. If $x_0 = \{1/n\}$, then $\mathfrak{M}(x_0) = (c_0)$. Sobczyk [19] showed there is no bounded projection of (m) on (c_0).

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